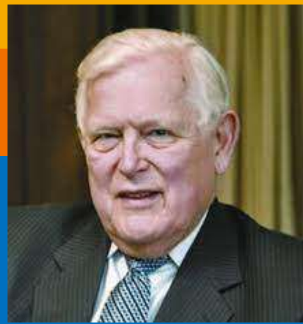
國立高雄科技大學
National Kaohsiung University of Science and Technology

Introduction to Kalman Filter



Rudolf Emil Kalman

Journal of Basic Engineering, 1960

Speaker: Shih-Shinh Huang

January 30, 2022

Rudolf Emil Kalman, “A New Approach to Linear Filtering and Prediction Problems”, *Transactions of the ASME, Journal of Basic Engineering,* 1960

Outline

- Introduction
  - About Kalman Filter
  - Idea of Kalman Filter
- Gaussian Distribution
  - Standard Gaussian
  - Multivariant Gaussian
  - Product of Two Gaussians
- Kalman Filter Algorithm
  - Problem Statement
  - Algorithm Overview
  - Prediction Stage
  - Update Stage

Introduction

- About Kalman Filter
  - Kalman filter is used to estimate **states** from the **measurements** in a dynamic system.
  - Kalman filter has demonstrated its usefulness in various applications.
    - visual object tracking
    - robot/vehicle navigation
    - data prediction task

Introduction

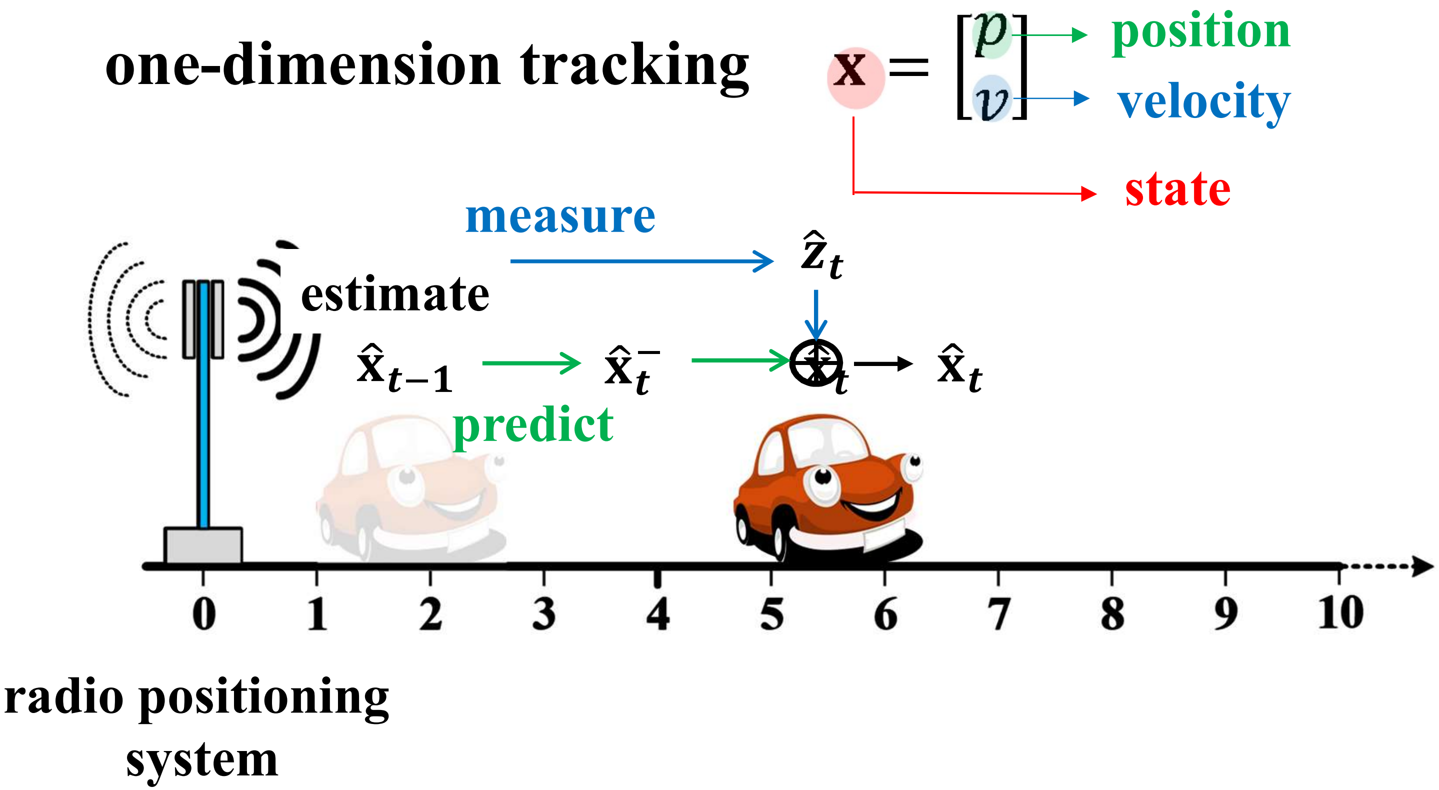
- About Kalman Filter
  - estimate the states in a **recursive** manner
    - very fast: without reprocessing all data at each time
    - light on memory: without storing all previous data
  - provide **good** estimates from the measurements with uncertainty
    - linear system: optimal estimate
    - non-linear system: qualified estimate

Introduction

- Idea of Kalman Filter
  - fuse the estimates from two sources that are with **uncertainty** (inaccuracy).
    - prediction: evolve from the estimate at previous time
    - measurement: perceive system state by sensors.
  - provide the estimate with less uncertainty than both from two sources.

Introduction

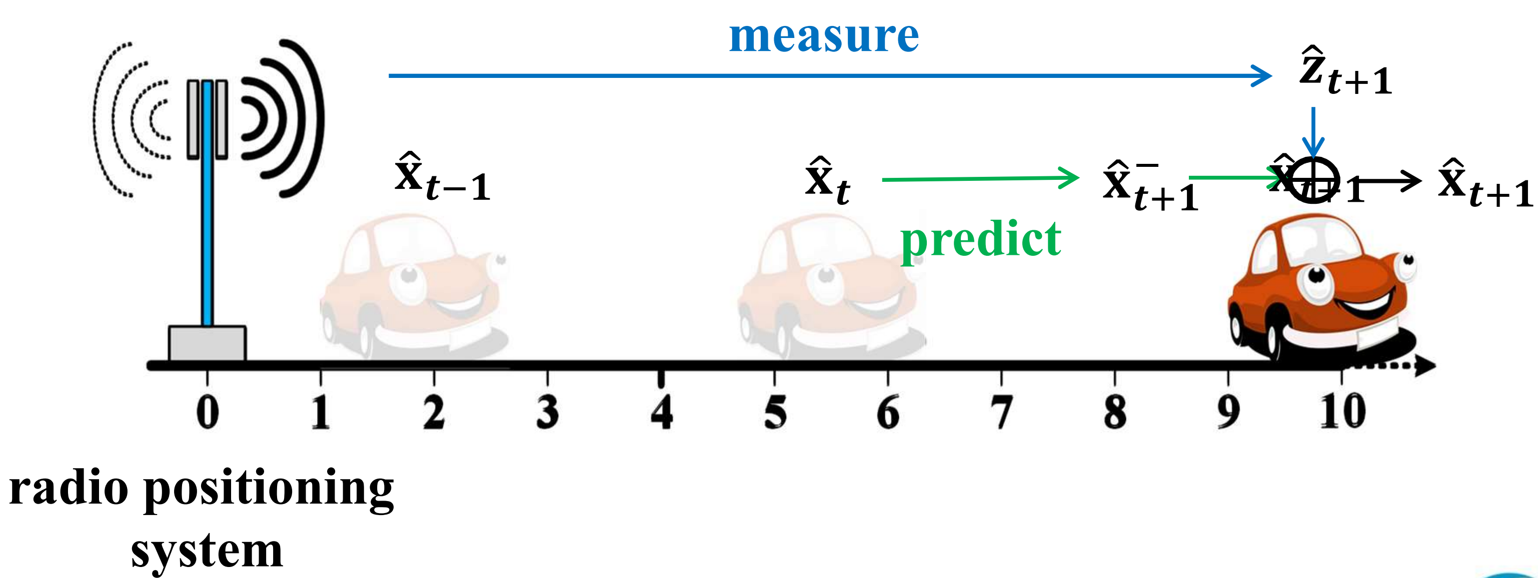
- Idea of Kalman Filter



Introduction

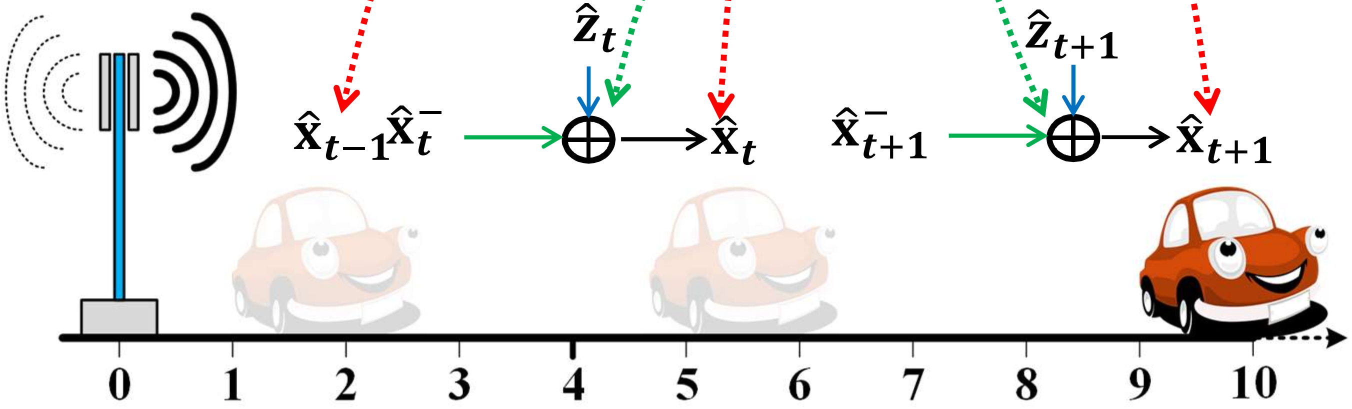
- Idea of Kalman Filter

one-dimension tracking $\mathbf{x} = \begin{bmatrix} p \\ v \end{bmatrix}$ $p \rightarrow$ **position**, $v \rightarrow$ **velocity**



Introduction

- Idea of Kalman Filter
  - estimate modeling: Gaussian distribution
  - estimate fusion: product of two Gaussians.



Gaussian Distribution

- Standard Gaussian $\eta(\mu, \sigma^2)$
  - A **random variable** x is Gaussian ($x \sim \eta(\mu, \sigma^2)$)

$$\Pr(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \times \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

$$\Pr(x) = \frac{1}{\sqrt{2\pi \times 0.67}} \times \exp\left(-\frac{(x-3.0)^2}{2 \times 0.67}\right)$$

mean

variance

three samples of x

$\boldsymbol{x_1 = 3.0}$

$\boldsymbol{x_2 = 4.0}$

$\boldsymbol{x_3 = 2.0}$

$$\mu = \frac{1}{3} \times (3.0 + 4.0 + 2.0) = 3.0$$

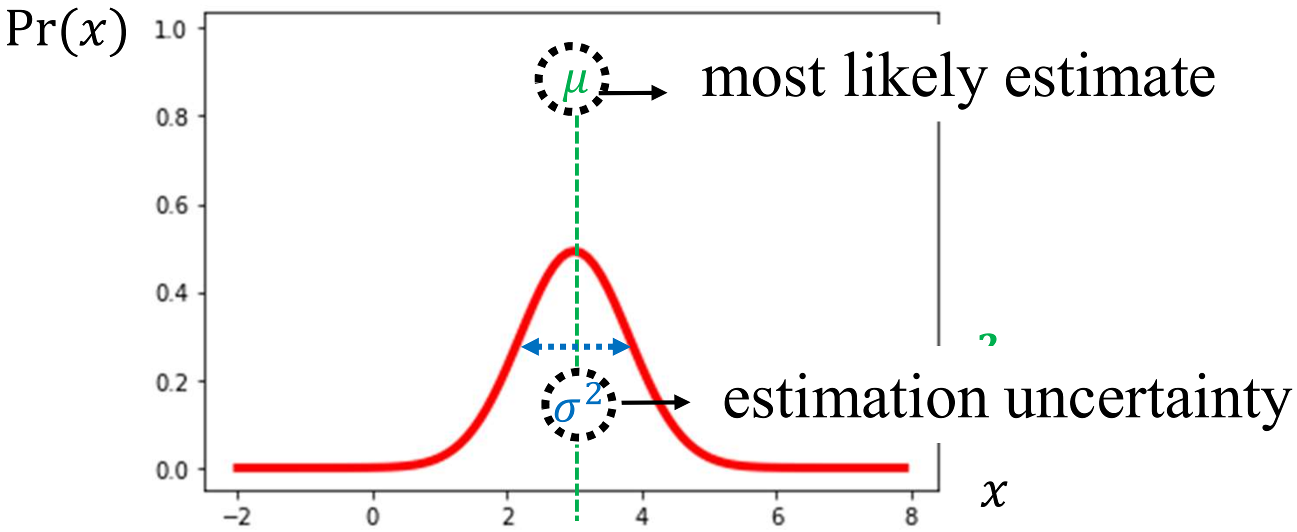
$$\sigma^2 = \frac{1}{3} \times \begin{pmatrix} (x_1 - \mu)^2 + \\ (x_2 - \mu)^2 + \\ (x_3 - \mu)^2 \end{pmatrix} \approx 0.67$$

$$\sigma^2 = \frac{1}{3} \times \begin{pmatrix} (3.0 - 3.0)^2 + \\ (4.0 - 3.0)^2 + \\ (2.0 - 3.0)^2 \end{pmatrix} \approx 0.67$$

Gaussian Distribution

- Standard Gaussian $\eta(\mu, \sigma^2)$

$$\Pr(x) = \frac{1}{\sqrt{2\pi \times 0.67}} \times \exp\left(-\frac{(x - 3.0)^2}{2 \times 0.67}\right)$$



(μ, σ^2) models an estimate of a random variable x

Gaussian Distribution

- Multivariant Gaussian $\eta(\boldsymbol{\mu}, \boldsymbol{\Sigma})$
  - A n-D **random vector** $\mathbf{x}$ is Gaussian ($\mathbf{x} \sim \eta(\boldsymbol{\mu}, \boldsymbol{\Sigma})$)

$$\Pr(\mathbf{x}) = \frac{1}{(2\pi)^{n/2} \times |\boldsymbol{\Sigma}|^{1/2}} \times \exp\left(-\frac{1}{2} \times (\mathbf{x} - \boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1} (\mathbf{x} - \boldsymbol{\mu})\right)$$

mean vector

covariance matrix

three samples of 2-D $\mathbf{x}$

$$\mathbf{x}_1 = \begin{bmatrix} 1.0 \\ 2.0 \end{bmatrix} \qquad \boldsymbol{\mu} = \frac{1}{3} \times \left\{ \begin{bmatrix} 1.0 \\ 2.0 \end{bmatrix} + \begin{bmatrix} 0.0 \\ 3.0 \end{bmatrix} + \begin{bmatrix} 2.0 \\ 1.0 \end{bmatrix} \right\} = \begin{bmatrix} 1.0 \\ 2.0 \end{bmatrix}$$

$$\mathbf{x}_2 = \begin{bmatrix} 0.0 \\ 3.0 \end{bmatrix}$$

$$\mathbf{x}_3 = \begin{bmatrix} 2.0 \\ 1.0 \end{bmatrix}$$

$$\boldsymbol{\Sigma} = \frac{1}{3} \times \left(\begin{array}{c} (\mathbf{x}_1 - \boldsymbol{\mu})(\mathbf{x}_1 - \boldsymbol{\mu})^T + \\ (\mathbf{x}_2 - \boldsymbol{\mu})(\mathbf{x}_2 - \boldsymbol{\mu})^T + \\ (\mathbf{x}_3 - \boldsymbol{\mu})(\mathbf{x}_3 - \boldsymbol{\mu})^T \end{array} \right)$$

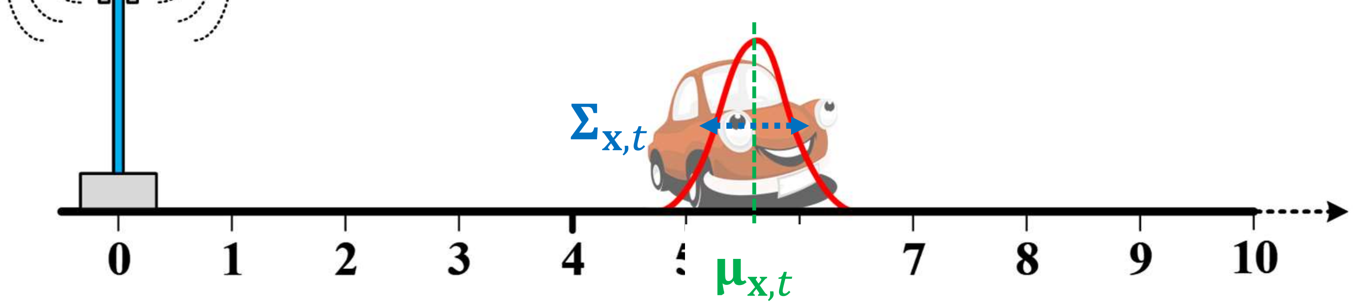
$$\boldsymbol{\Sigma} = \frac{1}{3} \times \left(\begin{bmatrix} 0.0 \\ 0.0 \end{bmatrix} \begin{bmatrix} 0.0 \\ 0.0 \end{bmatrix}^T + \begin{bmatrix} -1.0 \\ +1.0 \end{bmatrix} \begin{bmatrix} -1.0 \\ +1.0 \end{bmatrix}^T + \begin{bmatrix} +1.0 \\ -1.0 \end{bmatrix} \begin{bmatrix} +1.0 \\ -1.0 \end{bmatrix}^T \right)$$

$$\boldsymbol{\Sigma} = \begin{bmatrix} +\frac{2}{3} & -\frac{2}{3} \\ -\frac{2}{3} & +\frac{2}{3} \end{bmatrix}$$

Gaussian Distribution

- Multivariant Gaussian $\eta(\boldsymbol{\mu}, \boldsymbol{\Sigma})$
  - $(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ models an estimate $\hat{\mathbf{x}}$ of a random vector $\mathbf{x}$
    - $\boldsymbol{\mu}$: most likely estimate of $\mathbf{x}$
    - $\boldsymbol{\Sigma}$: uncertainty of $\mathbf{x}$

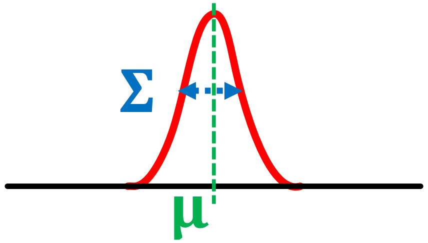
$\mathbf{x} = \begin{bmatrix} p \\ v \end{bmatrix}$ $\quad$ $\hat{\mathbf{x}}_t \sim (\boldsymbol{\mu}_{\mathbf{x},t}, \boldsymbol{\Sigma}_{\mathbf{x},t})$



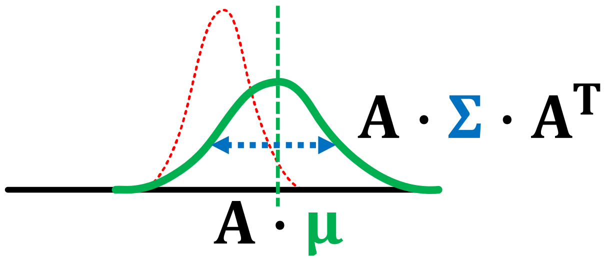
Gaussian Distribution

- Multivariant Gaussian $\eta(\boldsymbol{\mu}, \boldsymbol{\Sigma})$
  - multiplication formula: multiplying $\mathbf{x}$ by $\mathbf{A}$

$\mathbf{x} \sim \eta(\boldsymbol{\mu}, \boldsymbol{\Sigma})$

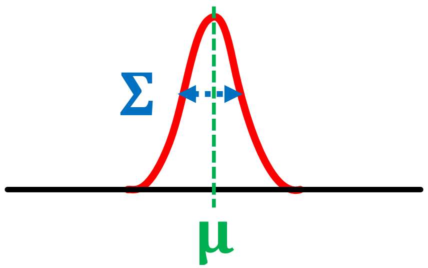


$\mathbf{A} \cdot \mathbf{x} \sim \eta(\mathbf{A} \cdot \boldsymbol{\mu},\ \mathbf{A} \cdot \boldsymbol{\Sigma} \cdot \mathbf{A}^{\mathrm{T}})$

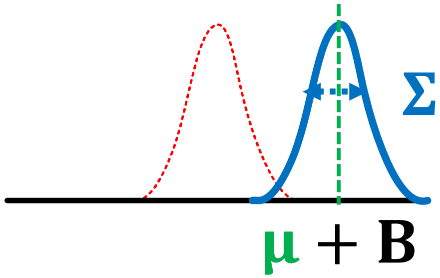


  - addition formula: adding $\mathbf{B}$ to $\mathbf{x}$

$\mathbf{x} \sim \eta(\boldsymbol{\mu}, \boldsymbol{\Sigma})$



$\mathbf{x} + \mathbf{B} \sim \eta(\boldsymbol{\mu} + \mathbf{B},\ \boldsymbol{\Sigma})$



Gaussian Distribution

- Product of Two Gaussians
  - Let x and y be two Gaussian random variables.

    $$x \sim \eta(\mu_x, \sigma_x^2) \qquad y \sim \eta(\mu_y, \sigma_y^2)$$

  - The product of x and y is a **scaled Gaussian**.

    $$xy \sim \lambda \times \eta(\mu_{xy}, \sigma_{xy}^2)$$

    $$\mu_{xy} = \frac{\mu_x \sigma_y^2 + \mu_y \sigma_x^2}{\sigma_x^2 + \sigma_y^2} \qquad \sigma_{xy}^2 = \frac{\sigma_x^2 \sigma_y^2}{\sigma_x^2 + \sigma_y^2}$$

Gaussian Distribution

- Product of Two Gaussians
  - Property: multiplying two Gaussians results in a **smaller** variance σ_{xy}^2.

proof: $\sigma_{xy}^2 \le \sigma_x^2$

$$\sigma_{xy}^2 - \sigma_x^2 = \frac{\sigma_x^2\sigma_y^2}{\sigma_x^2+\sigma_y^2} - \sigma_x^2$$

$$= \frac{\sigma_x^2\sigma_y^2 - (\sigma_x^4 + \sigma_x^2\sigma_y^2)}{\sigma_x^2+\sigma_y^2} = \frac{-\sigma_x^4}{\sigma_x^2+\sigma_y^2} \le 0$$

proof: $\sigma_{xy}^2 \le \sigma_y^2$

$$\sigma_{xy}^2 - \sigma_y^2 = \frac{\sigma_x^2\sigma_y^2}{\sigma_x^2+\sigma_y^2} - \sigma_y^2$$

$$= \frac{-\sigma_y^4}{\sigma_x^2+\sigma_y^2} \le 0$$

Gaussian Distribution

$$\mu_{xy} = \frac{\mu_x\sigma_y^2 + \mu_y\sigma_x^2}{\sigma_x^2 + \sigma_y^2} = \frac{\{\mu_x(\sigma_x^2 \cancel{+} \sigma_y^2) + \sigma_x^2(\mu_y - \mu_x)\}}{\sigma_x^2 \cancel{+} \sigma_y^2}$$

$$= \mu_x + \overbrace{\frac{\sigma_x^2}{\sigma_x^2 + \sigma_y^2}}^{=k} \times (\mu_y - \mu_x) \Longrightarrow \mu_{xy} = \mu_x + \boldsymbol{k} \times (\mu_y - \mu_x)$$

$$\sigma_{xy}^2 = \frac{\sigma_x^2\sigma_y^2}{\sigma_x^2 + \sigma_y^2} = \frac{\{\sigma_x^2(\sigma_x^2 \cancel{+} \sigma_y^2) - \sigma_x^2\sigma_x^2\}}{\sigma_x^2 \cancel{+} \sigma_y^2}$$

$$= \sigma_x^2 - \overbrace{\frac{\sigma_x^2}{\sigma_x^2 + \sigma_y^2}}^{=k} \times \sigma_x^2 \Longrightarrow \sigma_{xy}^2 = \sigma_x^2 - \boldsymbol{k} \times \sigma_x^2$$

Gaussian Distribution

- Product of Two Gaussians

$$\mathbf{x} \sim \eta(\boldsymbol{\mu}_\mathbf{x}, \boldsymbol{\Sigma}_\mathbf{x}) \qquad \mathbf{y} \sim \eta(\boldsymbol{\mu}_\mathbf{y}, \boldsymbol{\Sigma}_\mathbf{y})$$

$$\Longrightarrow \quad \mathbf{xy} \sim \lambda \times \eta(\boldsymbol{\mu}_\mathbf{xy}, \boldsymbol{\Sigma}_\mathbf{xy})$$

$$\begin{array}{l} \mu_x \leftarrow \boldsymbol{\mu}_\mathbf{x} \\ \mu_y \leftarrow \boldsymbol{\mu}_\mathbf{y} \\ \sigma_x^2 \leftarrow \boldsymbol{\Sigma}_\mathbf{x} \\ \sigma_y^2 \leftarrow \boldsymbol{\Sigma}_\mathbf{y} \end{array} \Longrightarrow \left\{ \begin{array}{l} \mu_{xy} = \mu_x + k \cdot (\mu_y - \mu_x) \\ \sigma_{xy}^2 = \sigma_x^2 - k\,\sigma_x^2 \\ k = \sigma_x^2 \cdot (\sigma_x^2 + \sigma_y^2)^{-1} \end{array} \right.$$

$$\left\{ \begin{array}{l} \boldsymbol{\mu}_\mathbf{xy} = \boldsymbol{\mu}_\mathbf{x} + \mathbf{K}(\boldsymbol{\mu}_\mathbf{y} - \boldsymbol{\mu}_\mathbf{x}) \\ \boldsymbol{\Sigma}_\mathbf{xy} = \boldsymbol{\Sigma}_\mathbf{x} - \mathbf{K}\boldsymbol{\Sigma}_\mathbf{x} \\ \mathbf{K} = \boldsymbol{\Sigma}_\mathbf{x} (\boldsymbol{\Sigma}_\mathbf{x} + \boldsymbol{\Sigma}_\mathbf{y})^{-1} \end{array} \right.$$

inverse

Kalman Filter Algorithm

- Problem Statement
  - assumption: the system is **linearly dynamic**
    - **linear prediction process:** evolving $\hat{\mathbf{x}}_t$ from $\hat{\mathbf{x}}_{t-1}$

$$\hat{\mathbf{x}}_t = \mathbf{F}_t \cdot \hat{\mathbf{x}}_{t-1} + \mathbf{B}_t \cdot u_t$$

state transition matrix

control input matrix

control input vector

    - **linear measure process:** operating on $\hat{\mathbf{x}}_t$ to produce sensor reading $\hat{\mathbf{z}}_t$

$$\hat{\mathbf{z}}_t = \mathbf{H}_t \cdot \hat{\mathbf{x}}_t$$

measurement matrix

$\mathbf{F}_t$, $\mathbf{B}_t$, $\mathbf{u}_t$ and $\mathbf{H}_t$: known **system** parameters

Kalman Filter Algorithm

- Problem Statement
  - assumption: the system is with **uncertainty**
    - noisy control input

$$\hat{\mathbf{x}}_t = \mathbf{F}_t \cdot \hat{\mathbf{x}}_{t-1} + \mathbf{B}_t \cdot \boldsymbol{u}_t + \boldsymbol{w}_t \quad \sim\eta(0, \mathbf{Q})$$

    - noisy sensing

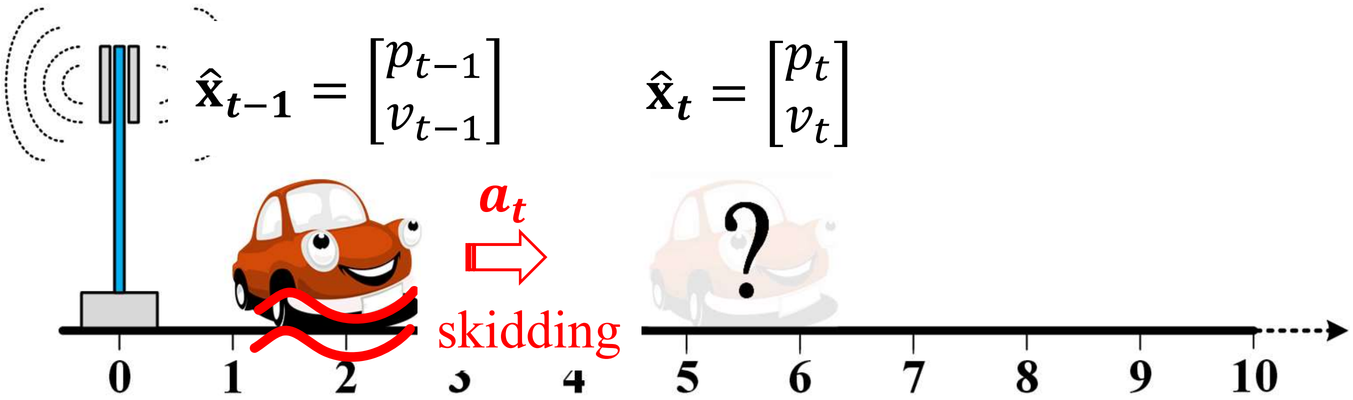
$$\hat{\mathbf{z}}_t = \mathbf{H}_t \cdot \hat{\mathbf{x}}_t + \boldsymbol{v}_t \quad \sim\eta(0, \mathbf{R})$$

zero-mean Gaussian

Q and **R** are the **tuning** parameters

Kalman Filter Algorithm

$\hat{\mathbf{x}}_{t-1} = \begin{bmatrix} p_{t-1} \\ v_{t-1} \end{bmatrix}$ $\hat{\mathbf{x}}_{t} = \begin{bmatrix} p_{t} \\ v_{t} \end{bmatrix}$



Kinematic Formula

⇨ time: t

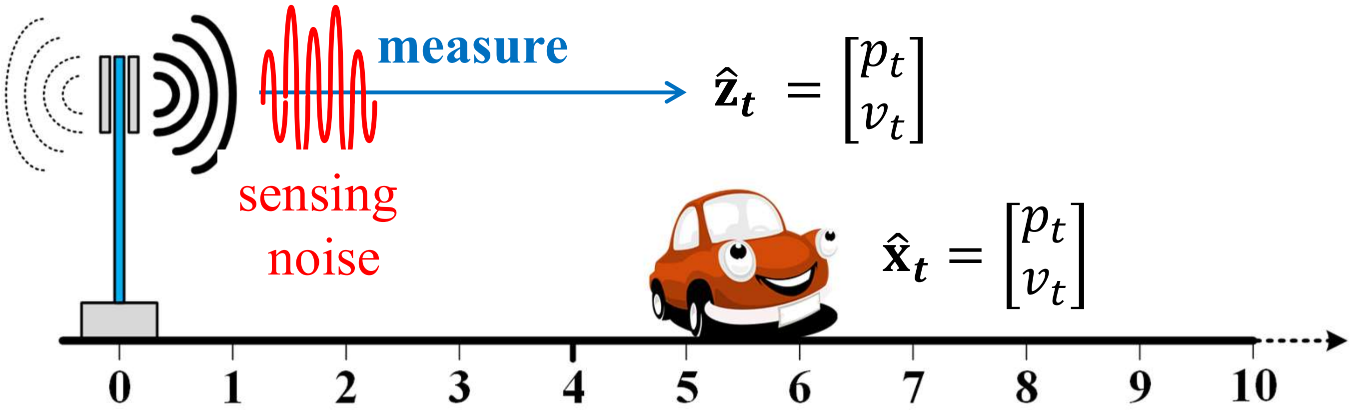
$$p_t = p_{t-1} + v_{t-1} \times \Delta t + \frac{1}{2} \times a_t \times \Delta t^2$$

$$v_t = v_{t-1} + a_t \times \Delta t$$

$$\begin{bmatrix} p_t \\ v_t \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & \Delta t \\ 0 & 1 \end{bmatrix}}_{\mathbf{F}_t} \begin{bmatrix} p_{t-1} \\ v_{t-1} \end{bmatrix} + \underbrace{\begin{bmatrix} \frac{\Delta t^2}{2} \\ \Delta t \end{bmatrix}}_{\mathbf{B}_t} \times \underbrace{a_t}_{\boldsymbol{u}_t}$$

$$\hat{\mathbf{x}}_t = \mathbf{F}_t \cdot \hat{\mathbf{x}}_{t-1} + \mathbf{B}_t \cdot \boldsymbol{u}_t + \boldsymbol{w}_t \sim \eta(0, \mathbf{Q})$$

Kalman Filter Algorithm



$$\hat{\mathbf{z}}_t = \begin{bmatrix} p_t \\ v_t \end{bmatrix}$$

$$\hat{\mathbf{x}}_t = \begin{bmatrix} p_t \\ v_t \end{bmatrix}$$

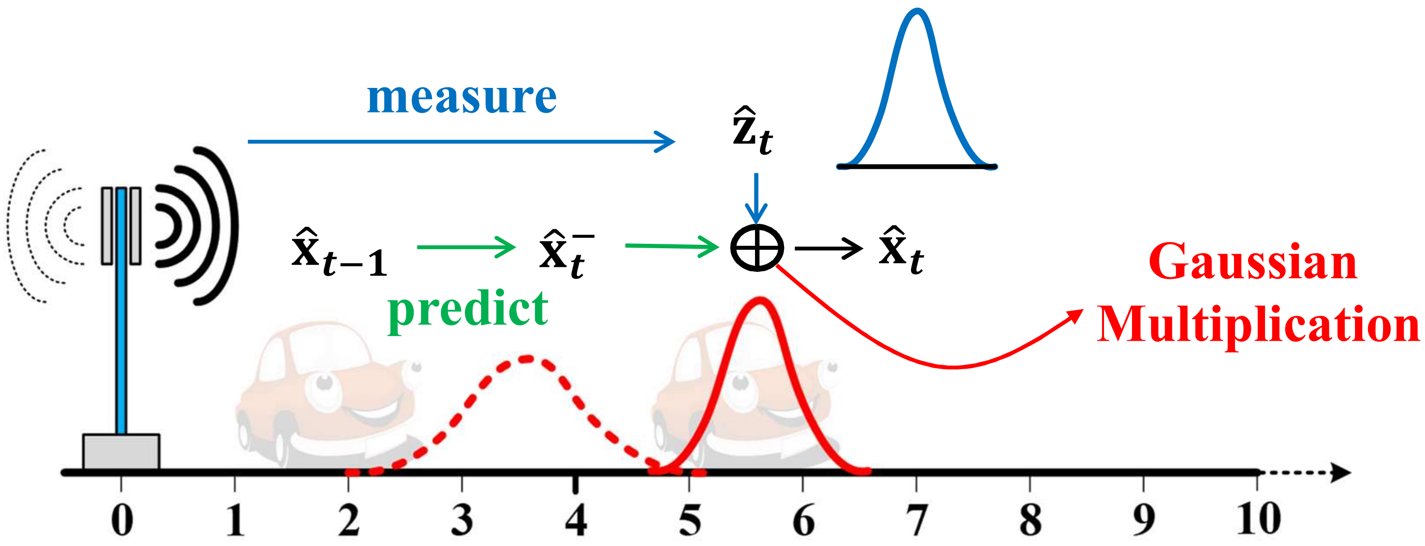
radio positioning system

$$\hat{\mathbf{z}}_t = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}_{\mathbf{H}_t} \hat{\mathbf{x}}_t$$

$$\hat{\mathbf{z}}_t = \mathbf{H}_t \cdot \hat{\mathbf{x}}_t + \boldsymbol{v}_t \quad \sim \eta(0, \mathbf{R})$$

Kalman Filter Algorithm

- Problem Statement
  - objective: obtain the estimate $\hat{\mathbf{x}}_t$ by fusing $\hat{\mathbf{x}}_t^-$ and $\hat{\mathbf{z}}_t$ Gaussian estimates



Kalman Filter Algorithm

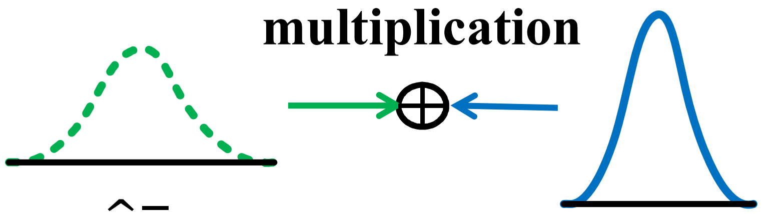
- Algorithm Overview
  - Input
    - $\hat{\mathbf{x}}_0 \sim \left(\boldsymbol{\mu}_{\mathbf{x},0}, \boldsymbol{\Sigma}_{\mathbf{x},0}\right)$: initial estimate
    - $\mathbf{Q}$ and $\mathbf{R}$: tuning parameters
    - $\{\boldsymbol{z}_1, \boldsymbol{z}_2, \ldots, \boldsymbol{z}_t\}$ **:** a sequence of measurements
  - Output
    - $\hat{\mathbf{x}}_t \sim \left(\boldsymbol{\mu}_{\mathbf{x},t}, \boldsymbol{\Sigma}_{\mathbf{x},t}\right)$: an estimate of the state $\mathbf{x}$ at time t

Kalman Filter Algorithm

- Algorithm Overview
  - prediction stage: predict $\hat{\mathbf{x}}_t^-$ at time t from $\hat{\mathbf{x}}_{t-1}$

$$\hat{\mathbf{x}}_t^- = \mathbf{F}_t \cdot \hat{\mathbf{x}}_{t-1} + \mathbf{B}_t \cdot \boldsymbol{u}_t + \boldsymbol{w}_t$$

  - update stage: fuse $\hat{\mathbf{x}}_t^-$ and $\hat{\mathbf{z}}_t$ by Gaussian product

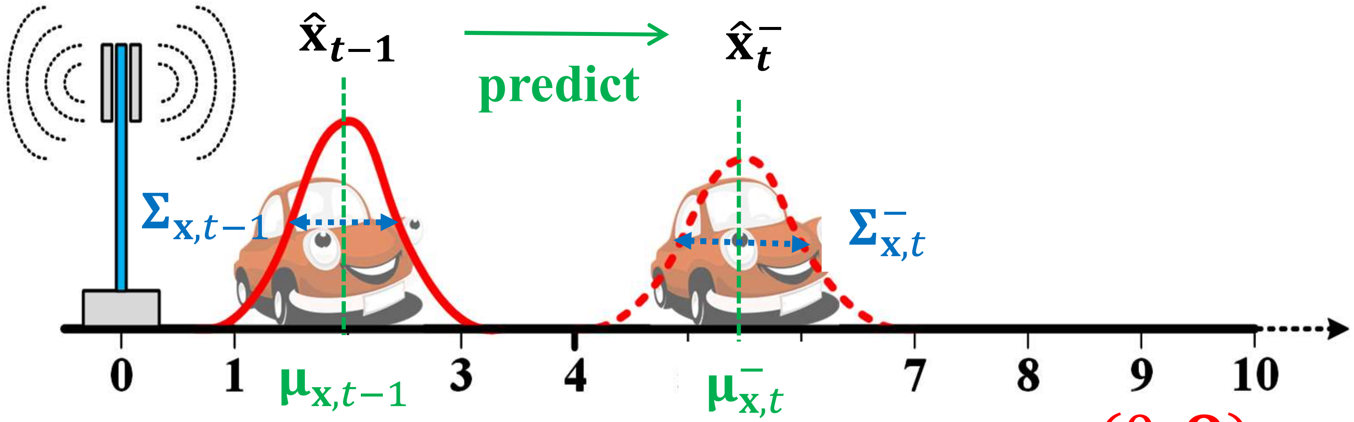


$\hat{\mathbf{x}}_t^-$

$$\hat{\mathbf{z}}_t = \mathbf{H}_t \cdot \hat{\mathbf{x}}_t + \boldsymbol{v}_t$$

Kalman Filter Algorithm

- Prediction Stage: $\hat{\mathbf{x}}_{t-1} \rightarrow \hat{\mathbf{x}}_t^-$



$$\mathbf{F}_t \cdot \hat{\mathbf{x}}_{t-1} + \mathbf{B}_t \cdot u_t + w_t \qquad \sim \eta(0, \mathbf{Q})$$

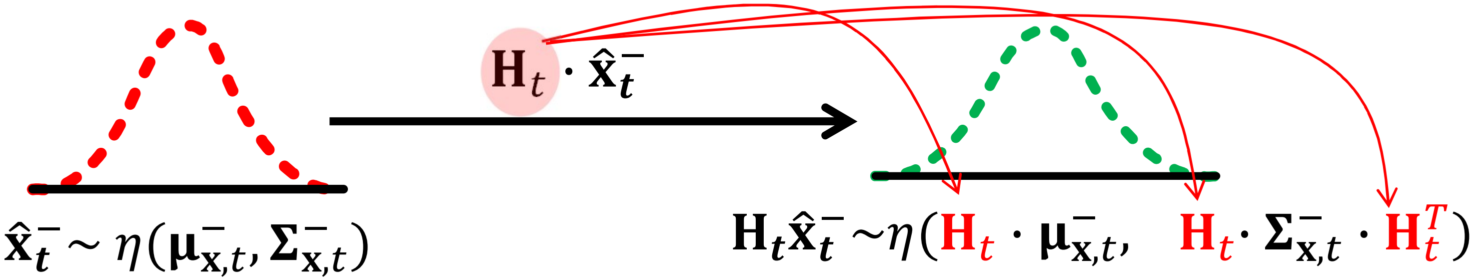
$$\eta(\mu_{\mathbf{x},t-1}, \Sigma_{\mathbf{x},t-1}) \rightarrow \eta(\mu_{\mathbf{x},t}^-, \Sigma_{\mathbf{x},t}^-)$$

$$\begin{cases} \mu_{\mathbf{x},t}^- = \mathbf{F}_t \cdot \mu_{\mathbf{x},t-1} + \mathbf{B}_t u_t \\ \Sigma_{\mathbf{x},t}^- = \mathbf{F}_t \cdot \Sigma_{\mathbf{x},t-1} \cdot \mathbf{F}_t^T + \mathbf{Q}_t \end{cases}$$

$$\eta(\mathbf{F}_t \cdot \mu_{\mathbf{x},t-1} + \mathbf{B}_t \cdot u_t , \mathbf{F}_t \cdot \Sigma_{\mathbf{x},t-1} \cdot \mathbf{F}_t^T + \mathbf{Q}_t)$$

Kalman Filter Algorithm

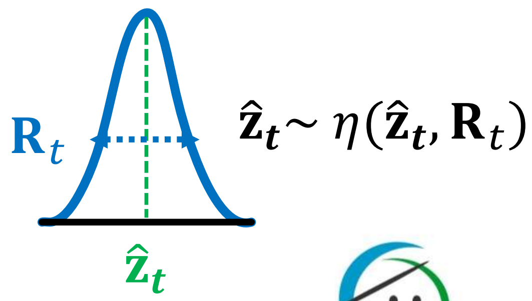
- Update Stage
  - transform $\hat{\mathbf{x}}_t^-$ to sensor-reading $\mathbf{z}$ space.



  - model the measurement as a Gaussian distribution

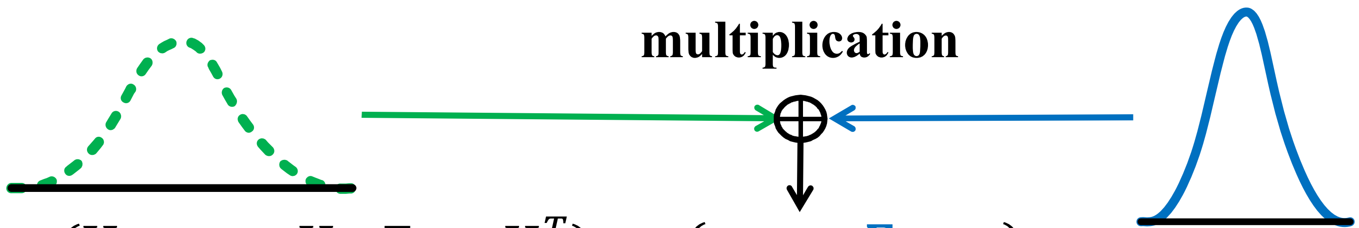
$$\hat{\mathbf{z}}_t = \mathbf{H}_t \cdot \hat{\mathbf{x}}_t + \boldsymbol{v}_t \sim \eta(0, \mathbf{R}_t)$$

- $\hat{\mathbf{z}}_t$: most likely estimate
- $\mathbf{R}_t$: measuring uncertainty



Kalman Filter Algorithm

- Update Stage
  - refine the estimate by multiplying two Gaussians



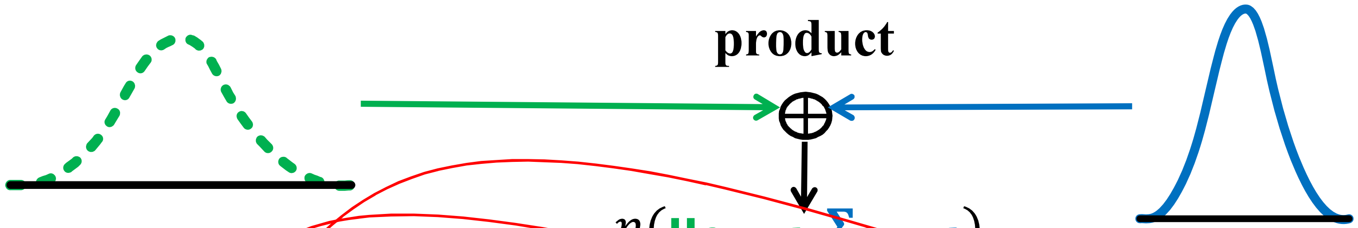
$\mathbf{H}_t\hat{\mathbf{x}}_t^- \sim \eta(\underbrace{\mathbf{H}_t \cdot \boldsymbol{\mu}_{\mathbf{x},t}^-}_{\boldsymbol{\mu}_\mathbf{x}},\ \underbrace{\mathbf{H}_t \cdot \boldsymbol{\Sigma}_{\mathbf{x},t}^- \cdot \mathbf{H}_t^T}_{\boldsymbol{\Sigma}_\mathbf{x}})$ $\quad \eta(\boldsymbol{\mu}_{\text{fused}}, \boldsymbol{\Sigma}_{\text{fused}})$ $\quad \hat{\mathbf{z}}_t \sim \eta(\underbrace{\hat{\mathbf{z}}_t}_{\boldsymbol{\mu}_\mathbf{y}}, \underbrace{\mathbf{R}_t}_{\boldsymbol{\Sigma}_\mathbf{Y}})$

$$\begin{cases} \boldsymbol{\mu}_{\text{fused}} = \boldsymbol{\mu}_\mathbf{x} + \mathbf{K} \cdot (\boldsymbol{\mu}_\mathbf{y} - \boldsymbol{\mu}_\mathbf{x}) \\ \boldsymbol{\Sigma}_{\text{fused}} = \boldsymbol{\Sigma}_\mathbf{x} - \mathbf{K}\boldsymbol{\Sigma}_\mathbf{x} \\ \mathbf{K} = \boldsymbol{\Sigma}_\mathbf{x} \cdot (\boldsymbol{\Sigma}_\mathbf{x} + \boldsymbol{\Sigma}_\mathbf{y})^{-1} \end{cases}$$

$$\begin{cases} \mathbf{H}_t \boldsymbol{\mu}_{\mathbf{x},t} = \mathbf{H}_t \boldsymbol{\mu}_{\mathbf{x},t}^- + \mathbf{K}(\hat{\mathbf{z}}_t - \mathbf{H}_t \cdot \boldsymbol{\mu}_{\mathbf{x},t}^-) \\ \mathbf{H}_t \boldsymbol{\Sigma}_{\mathbf{x},t} \cdot \mathbf{H}_t^T = \mathbf{H}_t \boldsymbol{\Sigma}_{\mathbf{x},t}^- \cdot \mathbf{H}_t^T - \mathbf{K} \cdot \mathbf{H}_t \cdot \boldsymbol{\Sigma}_{\mathbf{x},t}^- \cdot \mathbf{H}_t^T \\ \mathbf{K} = \mathbf{H}_t \cdot \boldsymbol{\Sigma}_{\mathbf{x},t}^- \cdot \mathbf{H}_t^T (\mathbf{H}_t \cdot \boldsymbol{\Sigma}_{\mathbf{x},t}^- \cdot \mathbf{H}_t^T + \mathbf{R}_t)^{-1} \end{cases}$$

Kalman Filter Algorithm

- Update Stage
  - determine the estimate $\hat{\mathbf{x}}_t$ in the state $\mathbf{x}$ space



$$\eta(\boldsymbol{\mu}_{\text{fused}}, \boldsymbol{\Sigma}_{\text{fused}})$$

$$\equiv \mathbf{H}_t\hat{\mathbf{x}}_t \sim \eta(\mathbf{H}_t \cdot \boldsymbol{\mu}_{\mathbf{x},t}, \mathbf{H}_t \cdot \boldsymbol{\Sigma}_{\mathbf{x},t} \cdot \mathbf{H}_t^T)$$

$$\begin{cases} \boldsymbol{\mu}_{\text{fused}} = \mathbf{H}_t \cdot \boldsymbol{\mu}_{\mathbf{x},t}^{-} + \mathbf{K} \cdot (\mathbf{z}_t - \mathbf{H}_t \cdot \boldsymbol{\mu}_{\mathbf{x},t}^{-}) \\ \boldsymbol{\Sigma}_{\text{fused}} = \mathbf{H}_t \cdot \boldsymbol{\Sigma}_{\mathbf{x},t}^{-} \cdot \mathbf{H}_t^T - \mathbf{K} \cdot \mathbf{H}_t \cdot \boldsymbol{\Sigma}_{\mathbf{x},t}^{-} \cdot \mathbf{H}_t^T \\ \mathbf{K} = \mathbf{H}_t \cdot \boldsymbol{\Sigma}_{\mathbf{x},t}^{-} \cdot \mathbf{H}_t^T \cdot (\mathbf{H}_t \cdot \boldsymbol{\Sigma}_{\mathbf{x},t}^{-} \cdot \mathbf{H}_t^T + \mathbf{R}_t)^{-1} \end{cases}$$

$$\mathbf{H}_t^{-1}$$

$$\cancel{\mathbf{H}_t}\cdot\boldsymbol{\mu}_{\mathbf{x},t} = \mathbf{H}_t\cdot\boldsymbol{\mu}^{-}_{\mathbf{x},t} + \mathbf{K}\cdot\left(\hat{\mathbf{z}}_t - \mathbf{H}_t\cdot\boldsymbol{\mu}^{-}_{\mathbf{x},t}\right)$$

$$\Rightarrow \quad \boldsymbol{\mu}_{\mathbf{x},t} = \overbrace{\mathbf{H}_t^{-1}\cdot\mathbf{H}_t}^{=\mathbf{I}}\cdot\boldsymbol{\mu}^{-}_{\mathbf{x},t} + \mathbf{H}_t^{-1}\cdot\mathbf{K}\cdot\left(\hat{\mathbf{z}}_t - \mathbf{H}_t\cdot\boldsymbol{\mu}^{-}_{\mathbf{x},t}\right)$$

$$\Rightarrow \quad \boldsymbol{\mu}_{\mathbf{x},t} = \boldsymbol{\mu}^{-}_{\mathbf{x},t} + \mathbf{H}_t^{-1}\cdot\mathbf{K}\cdot\left(\hat{\mathbf{z}}_t - \mathbf{H}_t\cdot\boldsymbol{\mu}^{-}_{\mathbf{x},t}\right)$$

$$\mathbf{H}_t^{-1}$$

$$\cancel{\mathbf{H}_t}\cdot\boldsymbol{\Sigma}_{\mathbf{x},t}\cdot\mathbf{H}_t^T = \cancel{\mathbf{H}_t}\cdot\boldsymbol{\Sigma}^{-}_{\mathbf{x},t}\cdot\mathbf{H}_t^T - \mathbf{K}\cdot\mathbf{H}_t\cdot\boldsymbol{\Sigma}^{-}_{\mathbf{x},t}\cdot\mathbf{H}_t^T$$

$$\Rightarrow \quad \boldsymbol{\Sigma}_{\mathbf{x},t}\cdot\cancel{\mathbf{H}_t^T} = \boldsymbol{\Sigma}^{-}_{\mathbf{x},t}\cdot\mathbf{H}_t^T - \mathbf{H}_t^{-1}\cdot\mathbf{K}\cdot\mathbf{H}_t\cdot\boldsymbol{\Sigma}^{-}_{\mathbf{x},t}\cdot\mathbf{H}_t^T$$

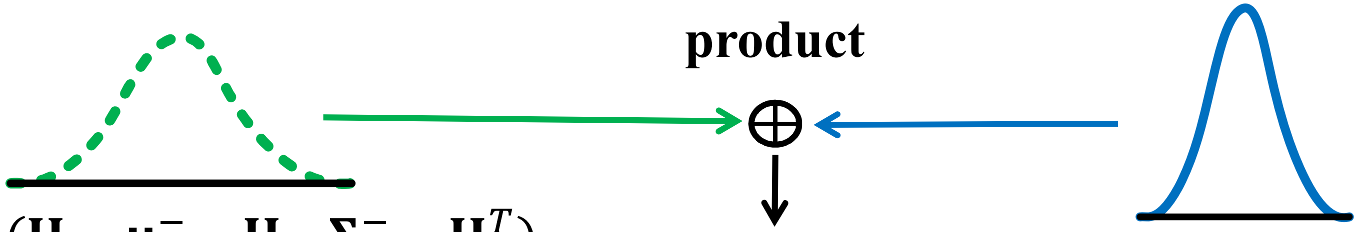
$$\left(\mathbf{H}_t^T\right)^{-1}$$

$$\Rightarrow \quad \boldsymbol{\Sigma}_{\mathbf{x},t} = \left(\boldsymbol{\Sigma}^{-}_{\mathbf{x},t}\cdot\cancel{\mathbf{H}_t^T} - \mathbf{H}_t^{-1}\cdot\mathbf{K}\cdot\mathbf{H}_t\cdot\boldsymbol{\Sigma}^{-}_{\mathbf{x},t}\cdot\cancel{\mathbf{H}_t^T}\right)\left(\mathbf{H}_t^T\right)^{-1}$$

$$\Rightarrow \quad \boldsymbol{\Sigma}_{\mathbf{x},t} = \boldsymbol{\Sigma}^{-}_{\mathbf{x},t} - \mathbf{H}_t^{-1}\cdot\mathbf{K}\cdot\mathbf{H}_t\cdot\boldsymbol{\Sigma}^{-}_{\mathbf{x},t}$$

Kalman Filter Algorithm

- Update Stage



$\mathbf{H}_t\hat{\mathbf{x}}_t^- \sim \eta(\mathbf{H}_t \cdot \boldsymbol{\mu}_{\mathbf{x},t}^-,\ \mathbf{H}_t \cdot \boldsymbol{\Sigma}_{\mathbf{x},t}^- \cdot \mathbf{H}_t^T)$ $\quad \hat{\mathbf{x}}_t \sim \eta(\boldsymbol{\mu}_{\mathbf{x},t}, \boldsymbol{\Sigma}_{\mathbf{x},t})$ $\quad \hat{\mathbf{z}}_t \sim \eta(\hat{\mathbf{z}}_t, \mathbf{R}_t)$

$$\begin{cases} \boldsymbol{\mu}_{\mathbf{x},t} = \boldsymbol{\mu}_{\mathbf{x},t}^- + \underbrace{\mathbf{K}_t}_{=\mathbf{K}} \left(\hat{\mathbf{z}}_t - \mathbf{H}_t \cdot \boldsymbol{\mu}_{\mathbf{x},t}^-\right) \\ \boldsymbol{\Sigma}_{\mathbf{x},t} = \boldsymbol{\Sigma}_{\mathbf{x},t}^- - \underbrace{\mathbf{K}_t \cdot \mathbf{H}_t}_{=\mathbf{K}} \cdot \boldsymbol{\Sigma}_{\mathbf{x},t}^- \\ \mathbf{K}_t = \underbrace{\boldsymbol{\Sigma}_{\mathbf{x},t}^- \cdot \mathbf{H}_t^T}_{=\mathbf{I}} \cdot \left(\mathbf{H}_t \cdot \boldsymbol{\Sigma}_{\mathbf{x},t}^- \cdot \mathbf{H}_t^T + \mathbf{R}_t\right)^{-1} \end{cases}$$

Kalman Gain

Kalman Filter Algorithm

t=1

$$\begin{cases} \boldsymbol{\mu}_{\mathbf{x},1}^{-} = \mathbf{F}_1 \cdot \boldsymbol{\mu}_{\mathbf{x},0} + \mathbf{B}_1 \boldsymbol{u}_1 \\ \boldsymbol{\Sigma}_{\mathbf{x},1}^{-} = \mathbf{F}_1 \cdot \boldsymbol{\Sigma}_{\mathbf{x},0} \cdot \mathbf{F}_1^T + \mathbf{Q}_1 \end{cases}$$

prediction stage

$\hat{\mathbf{x}}_0 \sim (\boldsymbol{\mu}_{\mathbf{x},0}, \boldsymbol{\Sigma}_{\mathbf{x},0})$

$\hat{\mathbf{x}}_1^{-} \sim (\boldsymbol{\mu}_{\mathbf{x},1}^{-}, \boldsymbol{\Sigma}_{\mathbf{x},1}^{-})$

$\hat{\mathbf{z}}_1 \sim \eta(\hat{\mathbf{z}}_1, \mathbf{R}_1)$

update stage

$$\begin{cases} \boldsymbol{\mu}_{\mathbf{x},1} = \boldsymbol{\mu}_{\mathbf{x},1}^{-} + \boldsymbol{\kappa} \cdot (\hat{\mathbf{z}}_1 - \mathbf{H}_1 \cdot \boldsymbol{\mu}_{\mathbf{x},1}^{-}) \\ \boldsymbol{\Sigma}_{\mathbf{x},1} = \boldsymbol{\Sigma}_{\mathbf{x},1}^{-} - \boldsymbol{\kappa} \cdot \mathbf{H}_1 \cdot \boldsymbol{\Sigma}_{\mathbf{x},1}^{-} \\ \boldsymbol{\kappa} = \boldsymbol{\Sigma}_{\mathbf{x},1}^{-} \cdot \mathbf{H}_1^T \cdot (\mathbf{H}_1 \cdot \boldsymbol{\Sigma}_{\mathbf{x},1}^{-} \cdot \mathbf{H}_1^T + \mathbf{R}_1)^{-1} \end{cases}$$

Kalman Filter Algorithm

t=2

$$\begin{cases} \boldsymbol{\mu}_{\mathbf{x},2}^{-} = \mathbf{F}_2 \cdot \boldsymbol{\mu}_{\mathbf{x},1} + \mathbf{B}_2 \boldsymbol{u}_2 \\ \boldsymbol{\Sigma}_{\mathbf{x},2}^{-} = \mathbf{F}_2 \cdot \boldsymbol{\Sigma}_{\mathbf{x},1} \cdot \mathbf{F}_2^{T} + \mathbf{Q}_2 \end{cases}$$

prediction stage

$\hat{\mathbf{x}}_2 \sim (\boldsymbol{\mu}_{\mathbf{x},2}, \boldsymbol{\Sigma}_{\mathbf{x},2})$

$\hat{\mathbf{x}}_2^{-} \sim (\boldsymbol{\mu}_{\mathbf{x},2}^{-}, \boldsymbol{\Sigma}_{\mathbf{x},2}^{-})$

$\hat{\mathbf{z}}_2 \sim \eta(\hat{\mathbf{z}}_2, \mathbf{R}_2)$

update stage

$$\begin{cases} \boldsymbol{\mu}_{\mathbf{x},2} = \boldsymbol{\mu}_{\mathbf{x},2}^{-} + \boldsymbol{\kappa} \cdot (\hat{\mathbf{z}}_2 - \mathbf{H}_2 \cdot \boldsymbol{\mu}_{\mathbf{x},2}^{-}) \\ \boldsymbol{\Sigma}_{\mathbf{x},2} = \boldsymbol{\Sigma}_{\mathbf{x},2}^{-} - \boldsymbol{\kappa} \cdot \mathbf{H}_2 \cdot \boldsymbol{\Sigma}_{\mathbf{x},2}^{-} \\ \boldsymbol{\kappa} = \boldsymbol{\Sigma}_{\mathbf{x},2}^{-} \cdot \mathbf{H}_2^{T} \cdot (\mathbf{H}_2 \cdot \boldsymbol{\Sigma}_{\mathbf{x},2}^{-} \cdot \mathbf{H}_2^{T} + \mathbf{R}_2)^{-1} \end{cases}$$


thank you
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